Oscillations in one page[*]

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$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega_d t). \tag{1}$$

Damped Oscillation

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0. \tag{2}$$

Characteristic polynomial of differential equation (2) is

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0.$$

This leads to $\lambda = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$. Therefore, there are three possible situations as shown below.

1. Underdamping(미급 감쇠)
   Let's define $\Omega = \sqrt{\omega_0^2 - \beta^2}$. Then
   $$x = c_1 e^{-\beta t} e^{i\Omega t} + c_2 e^{-\beta t} e^{-i\Omega t}. \tag{3}$$
2. Overdamping(과다 감쇠)
   Let's define $\Gamma = \sqrt{\beta^2 - \omega_0^2}$. Then
   $$x = c_1 e^{-\gamma_+ t} + c_2 e^{-\gamma_- t}, \tag{4}$$
   where $\gamma_\pm = \beta \pm \Gamma$.
   Note. since $\gamma_+ > \gamma_-$, $c_2 e^{-\gamma_- t}$ is dominant term in (4).
3. Critical damping(임계 감쇠)
   $$x = (c_1 + c_2 t)e^{-\beta t}. \tag{5}$$

By substituting initial conditions in (3), (4), or (5), we can get $x(t)$.

Forced Oscillation Solution of (1) can be expressed as $x = x_c + x_p$, where

$$m\ddot{x}_c + b\dot{x}_c + kx_c = 0, \tag{6}$$
$$m\ddot{x}_p + b\dot{x}_p + kx_p = F_0 \cos(\omega_d t). \tag{7}$$

The complimentary solution x_c converges to 0 when $t \to \infty$, since (6) is identical to (2).

Consider the test solution of (7) as $x_p(t) = A\cos(\omega t - \phi)$. By substituting this to (7), we get

$$x_p(t) = \frac{F_0/m}{\sqrt{(\omega_0^2 - \omega_d^2) + 4\omega_d^2\beta^2}} \cos\left(\omega_d t - \phi\right), \phi = \tan^{-1}\left(\frac{2\omega_d\beta}{\omega_0^2 - \omega_d^2}\right). \tag{8}$$

$$\omega_r = \sqrt{\omega_0^2 - 2\beta^2}. \qquad \text{(Amplitude resonance)}$$
$$\omega_K = \omega_0. \qquad \text{(Kinetic energy resonance)}$$

Question. Why ω_r is smaller than ω_0? Try to explain this phenomenon without any calculation.
Question. When $\beta > \omega_0/\sqrt{2}$, is amplitude resonance 'impossible'?

Q-factor : $Q \equiv \dfrac{w_r}{2\beta}$. Bigger Q implies more resonance.

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